Estimating parameters for a k-GIGARCH process

Abdou Kâ Diongue, Dominique Guegan

Estimating parameters of a k-factor GIGARCH process.

Estimation des paramètres d'un processus GIGARCH à k facteurs.

Abdou Kâ DIONGUE [a,b], Dominique GUEGAN [a]

[a] *ENS Cachan IDHE-MORA, UMR CNRS 8533, 61, avenue du président Wilson 94231 Cachan cedex. Tel: 01-47-40-55-75; Fax: 01-47-40-24-60.*

[b] *EDF R&D, 1 avenue du général de gaulle 92141 Clamart cedex. Tel: 01-47-65-53-61; Fax: 01-47-65-30-37.*

Abstract

Some crucial time series of market data, such as electricity spot prices, exhibit long-memory, in the sense of slowly-decaying correlations combined with heteroskedasticity. To be able to modelized such a behaviour, we consider in this Note the k-factor GIGARCH process and we propose two methods to address the related parameter estimation problem. For each method, we develop the asymptotic theory for estimation. *To cite this article: A.K. Diongue, D. Guégan C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Résumé

Plusieurs données de marché, telles que les prix spot de l'électricité, présentent de la longue mémoire, au sens de la décroissance hyperbolique des autocorrélations combinée avec un phénomène d'hétéroskédasticité. Pour modéliser de tels comportements, nous considérons dans cette Note les processus GIGARCH à k facteurs et nous proposons deux méthodes d'estimation des paramètres de ce modèle. Enfin, nous développons les propriétés asymptotiques de ces estimateurs. *Pour citer cet article : A.K. Diongue, D. Guégan C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Version française abrégée

Dans cette Note, nous nous intéressons à l'estimation des paramètres d'un processus GIGARCH à k facteurs par la méthode des moindres carrés conditionnels (CSS) et la méthode du maximum de vraisemblance de Whittle. Ce processus défini par les équations (1)-(2) a été introduit et étudié dans les articles

Email addresses: `abdou-ka.diongue@edf.fr` (Abdou Kâ DIONGUE), `guegan@ecogest.ens-cachan.fr` (Dominique GUEGAN).

de Guégan [7], [8]. Soit $\{X_t\}_{t=1}^T$ un processus GIGARCH à k facteurs stationnaire defini par les equations (1)-(2). Posons $\gamma = (\phi_1, \cdots, \phi_p, \theta_1, \cdots, \theta_q, d_1, \cdots, d_k)$, $\delta = (a_0, a_1, \cdots, a_r, b_1, \cdots, b_s)$ et $\omega = (\gamma, \delta)$. Supposons que $\omega_0 = (\gamma_0, \delta_0)$ soit la vraie valeur du paramètre ω et se trouve à l'intérieur du compact $\Theta \subseteq \mathbb{R}^{p+q+k+r+s+1}$. Nous supposons, dans toute la suite, que toutes les G-fréquences sont connues. Dans le Théorème 2.1, nous donnons les propriétés asymptotiques de l'estimateur des paramètres par la méthode CSS. Les propriétés asymptotiques des estimateurs des paramètres par la méthode de Whittle sont fournies dans le Théorème 2.2 pour les paramètres de mémoire longue et de mémoire courte homoscédastiques et dans le Théorème 2.3 pour les paramètres hétéroscédastiques.

Théorème 2.1 *Soit $\{X_t\}_{t=1}^T$ un processus generé par les équations (1)-(2). Supposons que $a_0 > 0$, $a_1, \cdots, a_r, b_1, \cdots, b_s \geq 0$, $\sum_{i=1}^{r} a_i + \sum_{i=1}^{s} b_i < 1$, $E\left(\varepsilon_t^4\right) < \infty$, $0 < d_i < \frac{1}{2}$ si $|\nu_i| < 1$, ou $0 < d_i < \frac{1}{4}$ si $|\nu_i| = 1$ pour $i = 1, \cdots, k$ et toutes les racines de $\phi(B)$ et $\theta(B)$ soient en dehors du cercle unité. Si les fréquences ν_i sont connues alors*

(i) *Il existe un estimateur CSS $\hat{\omega}_T$ satisfaisant $\frac{\partial L(\omega)}{\partial \omega} = 0$ et $\hat{\omega}_T \xrightarrow{P} \omega_0$ quand $T \to \infty$.*

(ii) $\sqrt{T}(\hat{\omega}_T - \omega_0) \xrightarrow{D} N\left(0, \Omega_0^{-1}\right)$ *quand $T \to \infty$, avec $\Omega_0 = diag\left(\Omega_{\gamma_0}, \Omega_{\delta_0}\right)$.*

(iii) *De plus, les estimateurs consistants des matrices d'information Ω_γ et Ω_δ sont donnés en (5).*

Théorème 2.2 *Soit $\{X_t\}_{t=1}^T$ un processus défini par les équations (1)-(2). Supposons vérifiées les hypothèses du Théorème (2.1). Alors, les estimateurs de Whittle des paramètres γ sont tels que :*

(i) $\hat{\gamma}_T \xrightarrow{p.s} \gamma_0$ *quand $T \to \infty$.*

(ii) $\sqrt{T}(\hat{\alpha}_T - \alpha_0) \xrightarrow{D} N\left(0, 4\pi V(\alpha_0)^{-1}\right)$, *quand $T \to \infty$, où $\alpha = (\phi_1, \cdots, \phi_p, \theta_1, \cdots, \theta_q)$, et où $V(\alpha)_{ij}$ est défini par l'équation (6).*

(iii) $\sqrt{T}\left(\hat{d}_T - d\right) \xrightarrow{D} N\left(0, 4\pi V(d)^{-1}\right)$, *où $V(d)_{ij}$ est donnée en (7)*

Théorème 2.3 *Soit $\{X_t\}_{t=1}^T$ un processus défini par les équations (1)-(2). Supposons vérifiées les hypothèses du Théorème (2.1). Alors*

(i) *Sous $(H_0)(J = 4)$ et (H_1), on a $\hat{\delta}_T \xrightarrow{P} \delta_0$, quand $T \to \infty$.*

(ii) *Sous $(H_0)(J = 8)$,(H_1) et (H_2), on a $\sqrt{T}\left(\hat{\delta}_T - \delta_0\right) \xrightarrow{D} N\left(0, 2W^{-1} + W^{-1}VW^{-1}\right)$, quand $T \to \infty$, où V est donné en (9)*

1. Introduction

Assume that $(\xi_t)_{t\in\mathbb{Z}}$ is a white noise process with unit variance and let the polynomials $\phi(B)$ and $\theta(B)$ denote the ARMA operators. Let B denote the backshift operator and $0 < d_i < \frac{1}{2}$ if $|\nu_i| < 1$ or $0 < d_i < \frac{1}{4}$ if $|\nu_i| = 1$ for $i = 1, \cdots, k$. We define a centered k-factor GIGARCH process $(X_t)_{t\in\mathbb{Z}}$ by, $\forall\, t$

$$\phi(B) \prod_{i=1}^{k} \left(I - 2\nu_i B + B^2\right)^{d_i} X_t = \theta(B)\, \varepsilon_t, \tag{1}$$

where

$$\varepsilon_t = \sqrt{h_t}\xi_t, \text{ with } h_t = a_0 + \sum_{i=1}^{r} a_i \varepsilon_{t-i} + \sum_{i=1}^{s} b_i h_{t-i}. \tag{2}$$

For $i = 1, \cdots, k$, the frequencies $\lambda_i = \arccos(\nu_i)$ are called the Gegenbauer frequencies (or G-frequencies). The process defined by the equations (1)-(2) was introduced by Guégan (see [7] and [8]). In the following section, we provide some results related to the asymptotic properties of the k-factor GIGARCH process estimators, obtained by two methods: the conditional sum of squares and the Whittle approach.

2. Asymptotic theory for estimation

Given a stationary k-factor GIGARCH process $\{X_t\}_{t=1}^{T}$ defined by the equations (1)-(2). We denote $\gamma = (\phi_1, \cdots, \phi_p, \theta_1, \cdots, \theta_q, d_1, \cdots, d_k)$, $\delta = (a_0, a_1, \cdots, a_r, b_1, \cdots, b_s)$ and $\omega = (\gamma, \delta)$ its parameters. We assume that $\omega_0 = (\gamma_0, \delta_0)$ is the true value of ω and is in the interior of the compact set $\Theta \subseteq \mathbb{R}^{p+q+k+r+s+1}$. Let us assume that all the G-frequencies are known.

2.1. *Conditional Sum of Squares estimation*

The conditional sum of squares estimator $\hat{\omega}_T$ of ω in Θ maximizes the conditional logarithmic likelihood $L(\omega)$ on F_0, where F_t is the σ-algebra generated by $(X_s, s \leq t)$. If we assume that the innovations $(\varepsilon_t)_{t\in\mathbb{Z}}$ have a conditional Gaussian distribution then the conditional log-likelihood is defined by:

$$L(\omega) = \frac{1}{T}\sum_{t=1}^{T} \ell_t, \quad \ell_t = -\frac{1}{2}\log(h_t) - \frac{\varepsilon_t^2}{2h_t}. \tag{3}$$

Now, if we assume that the innovations $(\varepsilon_t)_{t\in\mathbb{Z}}$ have a conditional Student distribution with l degrees of freedom, then the CSS estimator $\hat{\omega}_T$ maximizes the likelihood function $L(\omega)$ defined by

$$\begin{aligned} L(\omega) = T &\left[\log\Gamma\left\{\frac{(l+1)}{2}\right\} - \log\Gamma\left(\frac{l}{2}\right) - \frac{1}{2}\log(l-2)\right] \\ &- \frac{1}{2}\sum_{t=1}^{T}\left\{\log(h_t) + (l+1)\left[\log(1+\frac{\varepsilon_t^2}{h_t(l-2)})\right]\right\}. \end{aligned} \tag{4}$$

In the following Theorem, $L(\omega)$ represents the log likelihood introduced in (3) or in (4).

Theorem 2.1 *Suppose that the process $(X_t)_{t\in\mathbb{Z}}$ is generated by equations (1)-(2). Assume that $a_0 > 0, a_1, \cdots, a_r, b_1, \cdots, b_s \geq 0$, $\sum_{i=1}^{r} a_i + \sum_{i=1}^{s} b_i < 1$, $E(\varepsilon_t^4) < \infty$, $0 < d_i < \frac{1}{2}$ if $|\nu_i| < 1$ or $0 < d_i < \frac{1}{4}$ if $|\nu_i| = 1$ for $i = 1, \cdots, k$ and all roots of the polynomials $\phi(B)$ and $\theta(B)$ lie outside the unit circle. Then*

(i) There exists a CSS estimator $\hat{\omega}_T$ that satisfies $\frac{\partial L(\omega)}{\partial \omega} = 0$ and $\hat{\omega}_T \xrightarrow{P} \omega_0$ as $T \to \infty$.

(ii) $\sqrt{T}(\hat{\omega}_T - \omega_0) \xrightarrow{D} N(0, \Omega_0^{-1})$ as $T \to \infty$, where $\xrightarrow{P}$ and $\xrightarrow{D}$ denotes respectively the convergence in probability and in distribution. Furthermore, $\Omega_0 = diag(\Omega_{\gamma_0}, \Omega_{\delta_0})$ and Ω_{γ_0} and Ω_{δ_0} are values of Ω_γ and Ω_δ at $\omega = \omega_0$, with $\Omega_\gamma = E\left(\left[\frac{1}{h_t}\frac{\partial \varepsilon_t}{\partial \gamma}\frac{\partial \varepsilon_t}{\partial \gamma^T} + \frac{1}{2h_t^2}\frac{\partial h_t}{\partial \gamma}\frac{\partial h_t}{\partial \gamma^T}\right]\right)$ and $\Omega_\delta = E\left(\left[\frac{1}{2h_t^2}\frac{\partial h_t}{\partial \delta}\frac{\partial h_t}{\partial \delta^T}\right]\right)$.

(iii) The information matrices Ω_γ and Ω_δ can be estimated consistently by

$$\hat{\Omega}_\gamma = \frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{h_t}\frac{\partial \varepsilon_t}{\partial \gamma}\frac{\partial \varepsilon_t}{\partial \gamma^T} + \frac{1}{2h_t^2}\frac{\partial h_t}{\partial \gamma}\frac{\partial h_t}{\partial \gamma^T}\right] \text{ and } \hat{\Omega}_\delta = \frac{1}{T}\sum_{t=1}^{T}\left(\left[\frac{1}{2h_t^2}\frac{\partial h_t}{\partial \delta}\frac{\partial h_t}{\partial \delta^T}\right]\right). \tag{5}$$

The proof is given in Section 3 for the Gaussian case and details can be found in Diongue, Guégan and Vignal [4]. Note that, if the innovations $(\varepsilon_t)_{t\in\mathbb{Z}}$ have a conditional Student distribution with l degrees of freedom, then the proof can easily done using the same steps as in Section 3.

2.2. *Whittle estimation*

In this paragraph, we investigate the Whittle's method to estimate all parameters of the k-factor GIGARCH process defined by equations (1)-(2). The first step consists to estimate the long-memory parameters $d = (d_1, \cdots, d_k)$ and the ARMA(p,q) parameters $\alpha = (\phi_1, \cdots, \phi_p, \theta_1, \cdots, \theta_q)$ using the Whittle's approach (for more details, see Chung [2], [3] and Ferrara and Guégan chapter 8 of [5]). In the second step, the GARCH(r,s) parameters $\delta = (a_0, a_1, \cdots, a_r, b_1, \cdots, b_s)$ are estimated using Whittle's method applied to the residuals of the long-memory process (see Giraitis and Robinson [6] for more details).

Theorem 2.2 *Let $\{X_t\}_{t=1}^{T}$ be a k-factor GIGARCH process defined by equations (1)-(2). Let us assume that the same hypothesis given in Theorem (2.1) are verified. Then*

(i) $\hat{\gamma}_T \overset{a.s}{\to} \gamma_0$ as $T \to \infty$.

(ii) Furthermore: $\sqrt{T}\left(\hat{\alpha}_T - \alpha_0\right) \overset{D}{\to} N\left(0, 4\pi V\left(\alpha_0\right)^{-1}\right)$, as $T \to \infty$, Where,

$$V(\alpha)_{ij} = \int_{-\pi}^{\pi} g^2(\lambda,\omega)\frac{\partial g^{-1}(\lambda,\omega)}{\partial \alpha_i}\frac{\partial g^{-1}(\lambda,\omega)}{\partial \alpha_j}d\lambda. \tag{6}$$

Here $g(\lambda,\omega)$ denotes the spectral density of the process $(X_t)_{t\in\mathbb{Z}}$.

(iii) Moreover $\sqrt{T}\left(\hat{d}_T - d\right) \overset{D}{\to} N\left(0, 4\pi V(d)^{-1}\right)$, with

$$V(d)_{ij} = \int_{-\pi}^{\pi} \log\left|4\sin\left[\frac{(\lambda-\lambda_i)}{2}\right]\sin\left[\frac{(\lambda+\lambda_i)}{2}\right]\right| \log\left|4\sin\left[\frac{(\lambda-\lambda_j)}{2}\right]\sin\left[\frac{(\lambda+\lambda_j)}{2}\right]\right| d\lambda. \tag{7}$$

The Theorem 2.2 follows from the proof of Hosoya's Theorem 2.3 ([9]).

To estimate the GARCH(r,s) parameters δ, we consider the process $\left(\varepsilon_t^2\right)_{t\in\mathbb{Z}}$ in its ARMA representation. This means that we can rewritte (2) as: $\varepsilon_t^2 - \sum_{i=1}^{max(r,s)}(a_i + b_i)\varepsilon_{t-i}^2 = a_0 + \upsilon_t - \sum_{j=1}^{s} b_j \upsilon_{t-j}$, where $b_i = 0$ if $i \in (s,r]$ and $a_i = 0$ if $i \in (r,s]$. The process $(\upsilon_t)_{t\in\mathbb{Z}}$ defined by $\upsilon_t = \varepsilon_t^2 - h_t$ constitutes a white noise sequence with mean zero and variance σ^2. We introduce now some complementary assumptions to get the consistency and asymptotic normality of $\hat{\delta}_T$:

(H_0). For $t = 0, \pm 1, \ldots$, the process $(\xi_t)_{t\in\mathbb{Z}}$ introduced in equation (2), is strictly stationary, ergodic with finite Jth moment and $E(\xi_t|F_{t-1})=0$, $E(\xi_t^2|F_{t-1})=1$, and $E(\xi_t^{2j}|F_{t-1})=\upsilon_{2j}$ almost-surely, with $j=2,\cdots,\frac{J}{2}$, where υ_{2j} are constants such that $|\upsilon_J|^{\frac{2}{J}}\left(\sum_{i=1}^{r}a_i+\sum_{i=1}^{s}b_i\right)<1$.

(H_1).

(i) $\int_{-\pi}^{\pi}\log f(\lambda,\delta)\,d\lambda=0$, for all δ, with $f(\lambda,\delta)$ the spectral density of the process $(\varepsilon_t)_{t\in\mathbb{Z}}$.

(ii) $f(\lambda,\delta)^{-1}$ is continuous in $(\lambda,\delta)\in[-\pi,\pi]\times\Lambda$, where $\Lambda\subset\mathbb{R}^{r+s+1}$ is a compact.

(iii) $\mu_L(\{\lambda; f(\lambda,\delta)\neq f(\lambda,\delta_0)\})\geq 0$, for $\delta\in\Lambda$ with μ_L the Lebesgue measure.

(H_2).

(i) δ_0 is an interior point of Λ and in a neighborhood of δ_0, $\frac{\partial f(\lambda,\delta)^{-1}}{\partial\delta}$ and $\frac{\partial^2 f(\lambda,\delta)^{-1}}{\partial\delta\partial\delta^T}$ exist and are continuous in λ and δ.

(ii) $\frac{\partial f(\lambda,\delta_0)^{-1}}{\partial\delta}$ is K-Lipchitzienne with $K>\frac{1}{2}$.

(iii) The matrix W given by

$$W=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial\log f(\lambda,\delta_0)}{\partial\delta}\frac{\partial\log f(\lambda,\delta_0)}{\partial\delta^T}d\lambda \tag{8}$$

is non singular.

Theorem 2.3 *Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary, causal and invertible process defined by the equations (1)-(2).*

(i) Under (H_0) with $J=4$ and (H_1), $\hat{\delta}_T\xrightarrow{P}\delta_0$, as $T\to\infty$.

(ii) Under (H_0) with $J=8$, (H_1) and (H_2), $\sqrt{T}\left(\hat{\delta}_T-\delta_0\right)\xrightarrow{D}N\left(0,2W^{-1}+W^{-1}VW^{-1}\right)$, as $T\to\infty$. Here V is given by

$$V=\frac{2\pi}{\sigma^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\partial f(\lambda,\delta_0)^{-1}}{\partial\delta}\frac{\partial f(\omega,\delta_0)^{-1}}{\partial\delta^T}h(\lambda,-\omega,\omega)\,d\lambda d\omega, \tag{9}$$

with $h(\lambda,\omega,\upsilon)=\frac{1}{8\pi^3}\sum_{j,k,l=-\infty}^{+\infty}e^{ij\lambda-ik\omega-il\upsilon}Cum(\varepsilon_0,\varepsilon_j,\varepsilon_k,\varepsilon_l)$, and Cum is the order four's cumulant for the process $(\varepsilon_t)_{t\in\mathbb{Z}}$.

The proof of Theorem 2.3 is similar to the proofs of Theorem 2.1 and Theorem 2.2 given in Giraitis and Robinson's ([6]).

3. Proof of the Theorem 2.1

Here, we assume that the process $(\varepsilon_t)_{t\in\mathbb{Z}}$ have a conditional Gaussian distribution. We will first show *(iii)*. The strict stationarity and ergodicity of the process $(X_t)_{t\in\mathbb{Z}}$ and $(\varepsilon_t)_{t\in\mathbb{Z}}$ imply the consistency of the information matrices $\hat{\Omega}_\gamma$ and $\hat{\Omega}_\delta$.

In order to proof *(ii)*, we need to check the following Basawa's conditions (see Basawa, Feign and Heyde [1]):

– $\frac{1}{T}\sum_{t=1}^{T}\frac{\partial \ell_t(\omega_0)}{\partial \omega} \xrightarrow{P} 0$,

– there exists a nonrandom positive definite matrix $M(\omega_0)$ such that for all $\epsilon > 0$,

$$\mathbb{P}\left(-\frac{1}{T}\sum_{t=1}^{T}\frac{\partial^2 \ell_t}{\partial \omega \partial \omega^T} \geq M(\omega_0)\right) > 1-\epsilon \text{ for all } T > T_1(\epsilon),$$

– there exists a constant $M < \infty$ such that $E\left|\frac{\partial^3 \ell_t(\omega)}{\partial \omega_i \partial \omega_j \partial \omega_k}\right| < M$ for all $\omega \in \Theta$, where ω_i is the ith component of ω.

From $\frac{\partial \ell_t}{\partial \gamma} = \frac{1}{2h_t}\left(\frac{\varepsilon_t^2}{h_t}-1\right)\frac{\partial h_t}{\partial \gamma} - \frac{\varepsilon_t}{h_t}\frac{\partial \varepsilon_t}{\partial \gamma}$ and $\frac{\partial \ell_t}{\partial \delta} = \frac{1}{2h_t}\left(\frac{\varepsilon_t^2}{h_t}-1\right)\frac{\partial h_t}{\partial \delta}$, we have $E\left(\frac{\partial \ell_t}{\partial \omega}\right)_{\omega=\omega_0} = 0$ and using the ergodic theorem, Basawa's first condition follows.
The matrix Ω_0 is definite positive and hence the second Basawa's condition holds.
Now the last conditions of Basawa is obtained by differentiating $\frac{\partial^2 \ell_t}{\partial \omega \partial \omega^T}$ and using $E\left(\frac{\partial^3 \varepsilon_t}{\partial d_i \partial d_j \partial d_k}\right)^2 < \infty$. Condition *(ii)* follows.
Using *(iii)* of Theorem 2.1, we get $\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\partial \ell_t}{\partial \omega}\frac{\partial \ell_t}{\partial \omega^T}\right)_{\omega=\omega_0} \xrightarrow{a.s} \Omega_0$. Let S_T defined by $S_T = \sum_{t=1}^{T}\left(b_0\frac{\partial \ell_t}{\partial \omega}\right)_{\omega=\omega_0}$, where b_0 is an arbitrary constant vector and $b_0 b_0^T \neq 0$. Then, S_T is a martingal with $\frac{1}{T}E\left(S_T^2\right) = b_0^T \Omega_0 b_0 > 0$. Now, using the strict stationarity and ergodicity of the process $(X_t)_{t\in\mathbb{Z}}$ and $(\varepsilon_t)_{t\in\mathbb{Z}}$, we get $\left[\frac{1}{T}E(S_T)\right]^{-1}\left[\frac{1}{T}E\left(S_T^2|F_{t-1}\right)\right] \xrightarrow{a.s} 1$. From the Central Limit Theorem of Stout [10], the asymptotic normality convergence of the CSS estimators is derived.
Remark 1 : For applications and more details see Diongue, Guégan and Vignal [4].

References

[1] Basawa, I. V., Feign, P. D. and Heyde, C. C. Asymptotic properties of maximum likelihood estimators for stochastic processes, Sankhya, Ser. A, 38 (1976), 259-270.

[2] Chung, C.-F. Estimating a generalized long-memory process, Journal of Econometrics, 73 (1996), 237-259.

[3] Chung, C.-F. A generalized fractionally integrated ARMA process, Journal of Time Series Analysis, 17 (1994), 111-140.

[4] Diongue, A.K., Guégan, D. and Vignal, B. Processus GIGARCH: Estimation et applications aux prix spot de l'électricité, Preprint MORA, 14 (2003).

[5] Ferrara, L. Guégan, D. Comparison of parameter estimation methods in cyclical long-memory time series, Development in Forecasts Combination and Portfolio Choice, C. Dunis and J. Timmermann, J. Wiley,(2001).

[6] Giraitis, L. Robinson, P.M. Whittle Estimation of ARCH models, Econometrics Theory, 17 (2001), 608-631.

[7] Guégan D. A new model: The k-factor GIGARCH process, Journal of Signal Processing, 4 (2000), 265-271.

[8] Guégan D. A prospective study of the k-factor Gegenbauer process with heteroscedastic errors and an application to inflation rates, Finance India, 17 (2003), 1-20.

[9] Hosoya, Y. A Limit theory for long-range dependence and statistical inference on related models, The Annals of Statistics, 25, (1997), 105-137.

[10] Stout, W. F. Almost sure convergence, New York: Academic press, (1974).